

XXX. *Hints relating to the Use which may be made of the Tables of natural and logarithmic Sines, Tangents, &c. in the numerical Resolution of adfectèd Equations.* By William Wales, F. R. S. and Master of the Royal Mathematical School in Christ's Hospital.

Read June 14, 1781.

THE first intimation that I can meet with relating to the use which may be made of the tables of sines, tangents, and secants, in resolving adfectèd equations, is in the latter part of the second volume of Professor SAUNDERSON'S Elements of Algebra, printed in 1741, after his decease. The professor there shews how to resolve those two cases which make the first and second of the following examples, by means of the tables; but it appears, from many circumstances, he was not aware that the third case could be resolved in the same manner. All the three forms were, however, resolved by the late Mr. ANTHONY THACKER, a very ingenious man, who died in the beginning of the year 1744, by the help of a set of tables, of his own invention; different from, but in some measure analogous to, the tables of sines and tangents. These tables were finished and published, together with several papers concerning them, after his death, by a Mr. BROWN of Cleobury. In these papers, beside explaining fully the use of the tables in resolving cubic equations, Mr. THACKER shews that his method comprehends the resolution of all biquadratic equations, if they be

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first

first reduced to cubic ones in the manner which has been explained by DESCARTES and others, and the second term then taken away.

Since that time M. MAUDUIT has shewn how to find the roots of all the three forms of cubic equations, by means of the tables of sines, &c. in his excellent Treatise of Trigonometry. But none of these authors have attempted to resolve equations of more dimensions than three, by these means, without first reducing them to that number; nor even these, before the second term, or that which involves the square of the unknown quantity, is taken away: whereas such reductions will generally take up more time than is required to bring out the value of the unknown quantity by the following method; and, after all, frequently serve no other purpose but that of rendering the operation more intricate and troublesome.

The truly ingenious Mr. LANDEN, in his lucubrations, published in 1755, has given a general method of resolving that case of cubic equations, by means of the tables of sines, where all the roots are real, without the trouble of taking away the second term of the equation: and Mr. SIMPSON has shewn how to resolve equations of any dimensions, by the same means, provided those equations involve only the odd powers of the unknown quantity, and that the co-efficients observe such a law as will restrain the equation to that form which is expressive of the cosine of the multiple of an arc, of which the unknown quantity is the cosine. This was first done, I believe, by JOHN BERNOULLI, and afterwards by Mr. EULER, in his *Introduct. ad Analyt. Infinit.* and Mr. DE MOIVRE, in his *Miscell. Analyt.*; but the resolution of all equations of this form, as well as many others, is comprehended in the first of the following observations.

The first thought of extending the use of the tables of sines, tangents, and secants, farther than to the cases which have been already mentioned, occurred to me while I was considering the problem which produced the equation given in this paper as the fourth example. And it is remarkable, that the very same thought occurred to Dr. HURTON about the same time, and in the resolution of the same problem; and we were not a little surprized, on comparing our solutions together, to find that our ideas had taken so exactly the same turn; and that both should have stumbled on a thought, which, as far as either of us knew, had never presented itself to any one before. Having since examined farther into the matter, I have the satisfaction to find, that the principle is very extensive, and that a great number of equations, especially such as arise in the practice of geometry, astronomy, and optics, may be resolved by it with great ease and expedition.

But beside the facility with which the value of the unknown quantity is brought out by means of the tables of sines, tangents, and secants, this method of resolution has another considerable advantage over most others which have been proposed, inasmuch as the first state of the equation, without any previous reduction, is generally the best it can be in for resolution; and from which it may most readily be discovered, how to separate it into such parts as express the sine, or the tangent, or the secant of the arc of a circle; or into the sine, tangent, or secant of some multiple of that arc, or of a part of it; and in the doing of which consists the principal part of the business in question. It will also be of some advantage to preserve the original substitutions as distinct as possible, by using only the signs of the several operations which it may be necessary to go through

through in bringing the ſolution of a problem to an equation, inſtead of performing the operations themſelves.

Befides the advantages which this method of procedure affords to the mode of reſolution now more particularly under conſideration, it has ſo many others over that which is commonly made uſe of, that I am much ſurprized the latter ſhould ever have been adopted. By preſerving thus the original ſubſtitutions diſtinct, all the way through an operation, every expreſſion, even to the final equation, will exhibit the whole proceſs up to that ſtep; and it will appear as clearly, how every expreſſion has been derived, as it does in that mode of analyſis which was uſed by the ancient geometricians: whereas, when the ſeveral original expreſſions are melted down into one maſs by the multitude of actual additions, ſubtractions, multiplications, and diviſions, which they generally undergo, in a long algebraical proceſs, conducted in the uſual manner, it is impoſſible to trace the ſmalleſt veſtige of the original quantities in the final equation, except ſuch as are repreſented by a ſingle letter. Of courſe, however obvious the ſeveral ſteps might be at the time when they were taken, every idea of them muſt be totally loſt in the reſult; and it will be utterly impoſſible to trace them back again, in the manner they are done in the compoſition of a problem, the ſolution of which has been inveſtigated by the geometrical analyſis*. Let me add, that it is to this
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* This ſubject, if ever I am bleſſed with more leiſure than is at preſent my lot, ſhall be purſued farther in another paper: in which I ſhall endeavour to ſhew, that, notwithstanding the great difference which there appears to be between algebra and geometry, they are really but one ſcience, differently treated; and that the operations of the former may be rendered as clear and perſpicuous as thoſe of the latter are allowed to be. A diſquiſition of this nature will at leaſt have the merit of reſcuing a very uſeful and expeditious mode of inveſtigation
from

caufe we muft attribute all that obfcurity which the algebraic mode of inveftigation has been fo frequently charged with.

I fhall endeavour to verify this doctrine, in fome meafure, by the expreffions which are put down in the following tables for the fines, cofines, and tangents of arcs of circles, and of the multiples of thofe arcs; which tables will be found very ufe-ful in the profecution of the defign which I am now upon, and are abfolutely neceffary in the explanation of it.

from an unmerited ftigma: and if I never be happy enough to have an opportunity of doing it myfelf, what I have here faid may be the means of putting fome other perfon, who has, upon it.

T A B L E

Arc.	Sine.	Cofine.
A	x	$\sqrt{r+x} \cdot \overline{r-x}$
2A	$\frac{2x}{r} \cdot \sqrt{r+x} \cdot \overline{r-x}$	$\frac{r+x \cdot \overline{r-x} - x^2}{r}$
3A	$\frac{3x \cdot \overline{r+x} \cdot \overline{r-x} - x^3}{r^2}$	$\frac{r+x \cdot \overline{r-x} - 3x^2}{r^2} \sqrt{r+x} \cdot \overline{r-x}$
4A	$\frac{4x \cdot \overline{r+x} \cdot \overline{r-x} - 4x^3}{r^3} \sqrt{r+x} \cdot \overline{r-x}$	$\frac{r+x \cdot \overline{r-x} - 6x^2 \cdot \overline{r+x} \cdot \overline{r-x} + x^4}{r^3}$
5A	$\frac{5x \cdot \overline{r+x}^2 \cdot \overline{r-x}^2 - 10x^3 \cdot \overline{r+x} \cdot \overline{r-x} + x^5}{r^4}$	$\frac{r+x \cdot \overline{r-x}^2 \cdot \overline{r-x}^2 - 10x^2 \cdot \overline{r+x} \cdot \overline{r-x} + 5x^4}{r^4} \sqrt{r+x}$
6A	$\frac{6x \cdot \overline{r+x}^2 \cdot \overline{r-x}^2 - 20x^3 \cdot \overline{r+x} \cdot \overline{r-x} + 6x^5}{r^5} \sqrt{r+x} \cdot \overline{r-x}$	$\frac{r+x \cdot \overline{r-x}^3 \cdot \overline{r-x}^3 - 15x^2 \cdot \overline{r+x}^2 \cdot \overline{r-x}^2 + 15x^4 \cdot \overline{r+x}}{r^5}$

T A B L E

Arc.	Sine.	Cofine.
A	$\frac{rx}{\sqrt{r^2+x^2}}$	$r \cdot \frac{r}{\sqrt{r^2+x^2}}$
2A	$\frac{2r^2x}{r^2+x^2}$	$r \cdot \frac{r+x \cdot \overline{r-x}}{r^2+x^2}$
3A	$\frac{r+x \cdot \overline{r-x} + 2r^2}{r^2+x^2} \times \frac{rx}{\sqrt{r^2+x^2}}$	$\frac{r \cdot r+x \cdot \overline{r-x} - 2rx}{r^2+x^2} \times \frac{r}{\sqrt{r^2+x^2}}$
4A	$\frac{2 \cdot \overline{r+x} \cdot \overline{r-x}}{r^2+x^2} \times \frac{2r^2x}{r^2+x^2}$	$r \cdot \frac{r+x \cdot \overline{r-x} - 2rx}{r^2+x^2} \times \frac{r+x \cdot \overline{r-x} + 2rx}{r^2+x^2}$
5A	$\frac{r+x \cdot \overline{r-x}^2 + 4r^2 \cdot \overline{r+x} \cdot \overline{r-x} - 4r^2x^2}{r^2+x^2 \cdot r^2+x^2} \times \frac{rx}{\sqrt{r^2+x^2}}$	$r \cdot \frac{r+x \cdot \overline{r-x}^2 - 4x^2 \cdot \overline{r+x} \cdot \overline{r-x} - 4r^2x^2}{r^2+x^2 \cdot r^2+x^2} \times \frac{r+x \cdot \overline{r-x}^2 - 12r^2x^2}{r^2+x^2} \times \frac{r+x \cdot \overline{r-x}}{r^2+x^2} \times \frac{r}{r^2+x^2}$
6A	$\frac{3 \cdot \overline{r+x}^2 \cdot \overline{r-x}^2 - 4r^2x^2}{r^2+x^2 \cdot r^2+x^2} \times \frac{2 \cdot r^2x}{r^2+x^2}$	

E I.

	Tangent.
	$\frac{rx}{\sqrt{r+x} \cdot r-x}$
$\frac{-x}{-x+x^4}$	$\frac{2 \cdot \frac{rx}{r+x} \cdot r-x-x^2 \cdot \sqrt{r+x} \cdot r-x}{r+x \cdot r-x-x^2} \times \frac{rx}{\sqrt{r+x} \cdot r-x}$
$\frac{\sqrt{r+x} \cdot r-x}{4 \cdot r+x \cdot r-x-x^4}$	$\frac{3 \cdot \frac{rx}{r+x} \cdot r-x-x^2 \cdot \sqrt{r+x} \cdot r-x}{r+x \cdot r-x-3x^2} \times \frac{rx}{\sqrt{r+x} \cdot r-x}$
$\frac{4 \cdot r+x \cdot r-x-x^6}{r+x \cdot r-x-x^4}$	$\frac{4 \cdot \frac{rx}{r+x} \cdot r-x-4x^2 \cdot \sqrt{r+x} \cdot r-x}{r+x \cdot r-x-6x^2 \cdot r+x \cdot r-x+x^4} \times \frac{rx}{\sqrt{r+x} \cdot r-x}$
$\frac{5 \cdot r+x \cdot r-x-x^6}{r+x \cdot r-x-x^4}$	$\frac{5 \cdot \frac{rx}{r+x} \cdot r-x-10x^2 \cdot r+x \cdot r-x+x^4}{r+x \cdot r-x-10x^2 \cdot r+x \cdot r-x+5x^4} \times \frac{rx}{\sqrt{r+x} \cdot r-x}$
$\frac{6 \cdot r+x \cdot r-x-x^6}{r+x \cdot r-x-x^4}$	$\frac{6 \cdot \frac{rx}{r+x} \cdot r-x-20x^2 \cdot r+x \cdot r-x+6x^4}{r+x \cdot r-x-15x^2 \cdot r+x \cdot r-x+15x^4 \cdot r+x \cdot r-x} \times \frac{rx}{\sqrt{r+x} \cdot r-x}$

E II.

	Tangent.
	$\frac{x}{r+x \cdot r-x}$
$\frac{r}{+x^2}$	$\frac{2r^2x}{r+x \cdot r-x}$
$\frac{-x+2rx}{x^2}$	$\frac{x \cdot r+x \cdot r-x+2r^2x}{r+x \cdot r-x-2x^2}$
$\frac{r^2x^2}{\sqrt{r^2+x^2}} \times \frac{r}{r+x^2}$	$\frac{4r^2x \cdot r+x \cdot r-x}{r+x \cdot r-x-4r^2x^2}$
$\frac{-x}{r+x^2} \times \frac{r}{r+x^2}$	$\frac{x \cdot r+x \cdot r-x+4r^2x \cdot r+x \cdot r-x-4r^2x^3}{r+x \cdot r-x-4x^2 \cdot r+x \cdot r-x-4r^2x^2}$
	$\frac{r^2 \cdot \frac{6x \cdot r+x^2 \cdot r-x-8r^2x^3}{r+x \cdot r-x-12r^2x^2 \cdot r+x \cdot r-x}}{r+x \cdot r-x-12r^2x^2 \cdot r+x \cdot r-x}$

T A B L

Arc.	Sine.	Cofine.
A	$\frac{r \cdot \sqrt{x+r} \cdot x-r}{x}$	$r \cdot \frac{r}{x}$
2A	$r \cdot \frac{2r \cdot \sqrt{x+r} \cdot x-r}{x^2}$	$r \cdot \frac{r^2 - x + r \cdot x - r^2}{x^2}$
3A	$r \cdot \frac{3r^2 - x + r \cdot x - r}{x^3} \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^3 - 3r \cdot x + r \cdot x - r^3}{x^3}$
4A	$r \cdot \frac{4r^3 - 4r \cdot x + r \cdot x - r}{x^4} \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^4 - 6r^2 \cdot x + r \cdot x - r + x}{x^4}$
5A	$r \cdot \frac{5r^4 - 20r^2 \cdot x + r \cdot x - r + x + r]^2 \cdot x - r]^2}{x^5} \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^5 - 10r^3 \cdot x + r \cdot x - r + 5r \cdot x}{x^5}$
6A	$r \cdot \frac{6r^5 - 20r^3 \cdot r + x \cdot r - x + 6r \cdot x + r]^2 \cdot x - r]^2}{x^6} \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^6 - 15r^4 \cdot x + r \cdot x - r + 15r^2 \cdot x + r]^2}{x^6}$

T A B L

Arc.	Sine.	Cofine.
A	$\sqrt{r^2 - r - x}$	$r - x$
2A	$\frac{2 \cdot r - x}{r} \sqrt{r^2 - r - x}$	$\frac{2 \cdot r - x]^2 - r^2}{r}$
3A	$\frac{4 \cdot r - x]^2 - r^2}{r^2} \sqrt{r^2 - r - x}$	$\frac{4 \cdot r - x]^3 - 3r^2 \cdot r}{r^2}$
4A	$\frac{8 \cdot r - x]^3 - 4r^2 \cdot r - x}{r^3} \sqrt{r^2 - r - x}$	$\frac{8 \cdot r - x]^4 - 8r^2 \cdot r - x}{r^3}$
5A	$\frac{16 \cdot r - x]^4 - 12r^2 \cdot r - x]^2 + r^4}{x^4} \sqrt{r^2 - r - x}$	$\frac{16 \cdot r - x]^5 - 20r^2 \cdot r - x]^3}{r^4}$
6A	$\frac{32 \cdot r - x]^5 - 32r^2 \cdot r - x]^3 + 6r^4 \cdot r - x}{x^5} \sqrt{r^2 - r - x}$	$\frac{32 \cdot r - x]^6 - 48r^2 \cdot r - x]^4 + r}{r^5}$

tion of adfected Equations.

E III.

	Tangent.
$\sqrt{x+r} \cdot x-r$	$\sqrt{x+r} \cdot x-r$
$\frac{-r}{-r}$	$\frac{2r^2 \cdot \sqrt{x+r} \cdot x-r}{r^2 - x+r \cdot x-r}$
$\frac{-r}{-r}$	$\frac{3r^2 - x+r \cdot x-r}{r-3 \cdot x+r \cdot x-r} \sqrt{x+r} \cdot x-r$
$\frac{+r)^2 \cdot x-r)^2}{+r)^2 \cdot x-r)^2}$	$\frac{4r^4 - 4r^2 \cdot x+r \cdot x-r}{r^4 - 6r^2 \cdot x+r \cdot x-r + x+r)^2 \cdot x-r)^2} \sqrt{x+r} \cdot x-r$
$\frac{x+r)^2 \cdot x-r)^2}{x+r)^2 \cdot x-r)^2}$	$\frac{5r^4 - 10r^2 \cdot x+r \cdot x-r + x+r)^2 \cdot x-r)^2}{r^4 - 10r^2 \cdot x+r \cdot x-r + 5 \cdot x+r)^2 \cdot x-r)^2} \sqrt{x+r} \cdot x-r$
$\frac{\cdot x-r)^2 - x+r)^3 \cdot x-r)^3}{\cdot x-r)^2 - x+r)^3 \cdot x-r)^3}$	$\frac{6r^6 - 20r^4 \cdot x+r \cdot x-r + 6r^2 \cdot x+r)^2 \cdot x-r)^2}{r^6 - 15r^4 \cdot x+r \cdot x-r + 15r^2 \cdot x+r)^2 \cdot x-r)^2 - x+r)^3 \cdot x-r)^3} \sqrt{x+r} \cdot x-r$

E IV.

	Tangent.
	$\frac{r}{r-x} \sqrt{r^2 - r-x)^2}$
$\frac{-x}{-x}$	$\frac{2r \cdot r-x}{2 \cdot r-x)^2 - r^2} \sqrt{r^2 - r-x)^2}$
$\frac{r^2 + r^4}{r^2 + r^4}$	$\frac{4r \cdot r-x)^2 - r^3}{4 \cdot r-x)^3 - 3r^2 \cdot r-x} \sqrt{r^2 - r-x)^2}$
$\frac{+5r^4 \cdot r-x}{+5r^4 \cdot r-x}$	$\frac{8r \cdot r-x)^3 - 4r^3 \cdot r-x}{8 \cdot r-x)^4 - 8r^2 \cdot r-x)^2 + r^4} \sqrt{r^2 - r-x)^2}$
$\frac{8r^4 \cdot r-x)^2 - r^6}{8r^4 \cdot r-x)^2 - r^6}$	$\frac{16r \cdot r-x)^4 - 12r^3 \cdot r-x)^2 + r^5}{16 \cdot r-x)^5 - 20r^2 \cdot r-x)^3 + 5r^4 \cdot r-x} \sqrt{r^2 - r-x)^2}$
	$\frac{32r \cdot r-x)^5 - 32r^3 \cdot r-x)^3 + 6r^5 \cdot r-x}{32 \cdot r-x)^6 - 48r^2 \cdot r-x)^4 + 18r^4 \cdot r-x)^2 - r^6} \sqrt{r^2 - r-x)^2}$

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Observations on the foregoing tables.

EACH of the formulæ in these tables may be considered as one side of an equation, involving the unknown quantity x to different dimensions. In some of the formulæ the odd powers of x are only found, in others the even ones alone, and in others both; but they are all equally useful in finding the value of the unknown quantity in adfected equations which contain all the powers of that quantity, as will plainly appear from the following considerations.

I. If, on bringing the solution of any problem to an equation with some known quantity, it be found to correspond with any of the formula in these tables; or, if by any means it can be reduced to any of them, it is manifest, that nothing remains to be done but to divide the known side of the equation by the value of the quantity which is here denoted by r , and to seek for the quotient in the tables of sines, cosines, or tangents, as the case may require, and the value of the unknown quantity will be the sine, tangent, secant, or versed sine, of a given part of that arc (according as the expression is found in the first, second, third, or fourth table) multiplied by the value of r .

II. If, as will more frequently happen, the final equation of an operation be found equivalent to the sum, difference, product, or quotient, of some two or more of these formula; or to the sum, difference, product, or quotient, of some two or more of them multiplied or divided, increased or lessened, by some known quantity or quantities; then, having taken away

the known quantities by the common algebraic rules, observe the following ones.

1st. When the equation is found to correspond with the sum or difference of two formula in these tables, which are the sine and tangent, sine and cosine, or cosine and tangent, of the same arc, by running the eye along the tables of natural sines and tangents, find these two arcs, immediately following one another, the sum or difference of the sine and tangent, sine and cosine, or cosine and tangent, of which are one of them greater, and the other less than the number which constitutes the known side of the equation. Take the excess of one of these sums or differences above, and what the other sum or difference wants of the said given number, add these two errors together, and say, as the sum of them is to $60''$, so is that error which belongs to the less arc to a number of seconds; which being added to the less arc will give one, the sum or difference of whose sine and tangent, sine and cosine, or cosine and tangent, is exactly equal to the number which constitutes the known side of the equation. Of the arc, thus found, let such a part be taken as the table in which the formula are found directs, and the natural sine, tangent, secant, or versed sine (as the case may require) of this part, being multiplied by the value of r , if r be found in the equation, will be the value of x sought.

2d. When the equation happens to be the product or quotient of two formulæ which express the sine and cosine, sine and tangent, or cosine and tangent, of the same arc, take the logarithm of the number which constitutes the known side of the equation, and then follow exactly the directions given in the first case, using the tables of logarithmic sines and tangents instead of the tables of natural ones.

3d. If the equation, finally resulting from the resolution of any problem, present itself in an expression which is composed of the sum or difference of the sine, cosine, or tangent, of an arc, of which the unknown quantity is the sine, cosine, tangent, or versed sine, and the sine, cosine, or tangent, of some multiple of that arc, it will then be convenient to have two tables of sines and tangents; and in running the eye along them to find the two arcs immediately following one another, of which the sum or difference of the sine, cosine, or tangent, of one of them, and the sine, cosine, or tangent, of some multiple of it, may be less, and the sum or difference of the sine, cosine, or tangent, of the other, and the sine, cosine, or tangent, of the same multiple of it, may be greater than the number which constitutes the known side of the equation, for every minute of a degree that the finger is moved over in one, it must be moved over a number of minutes in the other, which is equal to the number of times that the single arc is contained in the multiple one. When these two arcs are found, the operation will not differ so materially from that which is pointed out in the first rule as to merit repetition.

4th. If, instead of the sum or difference of the sine, cosine, or tangent, of an arc, and the sine, cosine, or tangent, of some multiple of it, the form of the equation be such as to be constituted of the product of them, or the quotient of one divided by the other, the last rule will still hold good, using only the logarithmic sines and tangents instead of the natural ones, and comparing the sum or difference of them, according as the equation is composed of the product or quotient of the two factors, with the logarithm of the number which constitutes the known side of the equation, instead of that number itself.

5th. Sometimes the final equation will come out in expressions which are constituted of the sum, difference, product, or quotient, of the sine, cosine, or tangent, of some multiple of an arc, of which the unknown quantity is the sine, tangent, secant, or versed sine, and the sine, cosine, or tangent, of some other multiple of the same arc. And in any of these cases it is manifest, that the method of proceeding, in order to obtain one of the multiple arcs, and from thence the single one, of which the unknown quantity is the sine, tangent, &c. will not be greatly different from those which have been described in the third and fourth rules. The most material difference consists in this, that instead of proceeding minute by minute, according to the directions in the third rule to find the single arc, it will now be most convenient to proceed in each table by as many minutes at each step as are equal in number to the number of times which the single arc is contained in the multiple ones respectively.

6th. Equations will frequently make their appearance in formulæ which express the square, cube, &c. of the sine, cosine, or tangent, of the multiple of some arc, of which the unknown quantity is the sine, tangent, secant, or versed sine; or in formulæ which are expressive of the sum, difference, product, &c. of the sine, cosine, or tangent, of an arc, and some power of the sine, cosine, or tangent, of the same arc; or of some multiple of it, the unknown quantity being some other trigonometrical line belonging to that arc. Or the equation may be compounded of the sum, difference, product, &c. of the same, or different powers of the sines, tangents, or cosines, of different multiples of an arc, the unknown quantity being the sine, tangent, secant, or versed sine, of that arc. In every one of these cases the tables will give the value of the unknown quantity,

quantity, and in most of them with great ease and expedition. The method which is to be pursued in each case will readily present itself to a skilful analyst, who attends carefully to what has been already said, and to the examples which follow.

IV. The formula in the four preceding tables may be greatly varied by supposing x , the unknown quantity, to be some part or parts of the sine, tangent, &c. as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, &c. or some multiple of it, as twice, thrice, &c. Or x may be the square, or the square root, or any other power of the sine, tangent, secant, or versed sine, of an arc; in every one of which cases the formula will put on different appearances, either with respect to the powers or co-efficients of the unknown quantity, and yet admit of the same kind of application.

V. The tables may be rendered yet more extensively useful by inserting expressions for the sines, cosines, and tangents, of half the arc which has x for its sine, tangent, secant, or versed sine; and also for the sines, cosines, and tangents, of the odd multiples of this half arc, which expressions, together with those already inserted, may be considered as the sines, cosines, and tangents, of the multiples of an arc, the unknown quantity, being the sine, tangent, &c. of twice that arc. And this consideration may sometimes be applied to very useful purposes.

VI. In order to render the formula in the tables more general, I have put r for the radius of the circle; whereas it will frequently happen, that the equation, finally resulting from the resolution of a problem, especially those which relate to the doctrine of the sphere, will present itself in a form where the radius must be taken equal to unity: what these forms are will readily appear by substituting unity for r and its powers every where in the expression.

It would be endless were I to undertake to enumerate all the various circumstances and cases in which this method of bringing out the unknown quantity may be applied with success: what has already been said will be sufficient to explain the nature of it, and to enable the analyst to apply it in other instances as they occur to him, I shall therefore only add a few examples to illustrate it.

E X A M P L E I.

Let it be required to find the value of x in an equation of the form $x^3 - r^2x = a$.

If r^2 be expounded by 50, and a by 120 (see Phil. Trans. vol. LXVIII. p. 937.) the equation may be reduced to $\sqrt{x} \times \sqrt{x^2 - 50} = \sqrt{120}$; and, consequently, by tab. III. if x be considered as the secant of an arc, of which the radius is $\sqrt{50}$, $\sqrt{x^2 - 50}$ will be the tangent of it, and we shall have to find an arc, such that the tangent multiplied by the square root of the secant may be equal to $\sqrt{120}$; or, which amounts to the same thing, such an arc that the log. tang. together with half the log. secant may be equal to half the log. of 120. But because the tangent and secant, here required, are to the radius of the $\sqrt{50}$, the log. tangents and secants in the tables must be increased by the logarithm of that number, and therefore $\log. \text{tang.} + \frac{1}{2} \log. 50 + \frac{1}{2} \log. \text{secant} + \frac{1}{4} \log. 50 = \frac{1}{2} \log. 120$: or $\log. \text{tang.} + \frac{1}{2} \log. \text{secant} = \frac{1}{2} \log. 120 - \frac{3}{4} \log. \text{of } 50$. Hence, having taken $\frac{3}{4}$ the log. of 50 from $\frac{1}{2}$ the log. of 120, run the eye along the tables of logarithmic tangents and secants until an arc be found of which the sum of the log. tangent and half the log. secant is equal to 19.7653631, the remainder.

In

In this manner it will be readily found, that the sum of the log. tangent and half the log. secant of $28^{\circ} 37'$ is less than that difference by 2012, and that the sum of the log. tangent and half the log. secant of $28^{\circ} 38'$ is greater than it by 1337: therefore $3349 (2012 + 1337) : 60'' :: 2012 : 36''$. The exact arc, therefore, of which the sum of the log. tangent and half the log. secant is equal to 19.7653631 is $28^{\circ} 37' 36''$, and the log. secant of it is 10.0566242, which being increased by 0.8494850, the log. of $\sqrt{50}$ gives 0.9061092, which is the logarithm of 8.055810, the value of x sought, and which is true to seven places of figures.

E X A M P L E II.

To find the value of x in an equation of the form $x^3 - r^2x = -a$.

If r be expounded by 3, and a by 10, as they are in the example, given at p. 433. of the Phil. Transf. vol. LXX. the equation will be $x^3 - 9x = -10$, and may be transformed to $\sqrt{x} \times \sqrt{9-x^2} = \sqrt{10}$; and, therefore, by tab. I. the square root of the sine into the cosine of an arc, of which the radius is 3, is equal to the square root of 10. Consequently, an arc must be found, such that the sum of the log. cosine and half the log. tangent is equal to half the log. of 10. But because the radius of this arc must be 3, the log. sines and cosines must be increased by the log. of 3; and, therefore, log. cos. + log. of 3 + $\frac{1}{2}$ log. sine + $\frac{1}{2}$ log. of 3 must be equal to half the log. of 10; or, an arc must be found of which the sum of the tabular log. cosine and half the log. sine is equal to the difference between half the log. of 10 and $1\frac{1}{2}$ the log. of 3. Hence, having subtracted $1\frac{1}{2}$ log. of 3 from half the log. of 10, run the eye

2

along

along GARDINER's tables of logarithmic fines, by which means it will be readily found, that the sum of the log. cosine and half the log. sine of $28^{\circ} 53' 30''$ is less than 19.7843181, the excess of half the log. of 10 above $1\frac{1}{2}$ log. 3, by 15, and that the sum of the log. cosine and half the log. sine of $28^{\circ} 53' 40''$ is greater than that difference by 60. Consequently $75 (15 + 60) : 10'' :: 15 : 2''$. The exact arc, therefore, of which the sum of the log. cosine and half the log. sine is equal to 19.7843181, is $28^{\circ} 53' 32''$; and the log. sine of this arc, increased by the log. of 3, is 0.1612153, the logarithm of 1.44949, the value of x required, true to the last place.

But many equations of this form, and this example among the rest, admit of two positive values of the unknown quantity; and by carrying the eye farther along the tables it will be found also, that the sum of the log. cosine and half the log. sine of $41^{\circ} 48' 30''$ is greater than 19.7843181 by 50, and that the sum of the log. cosine and half the log. sine of $41^{\circ} 48' 40''$ is too little by 21. Consequently, $71 (50 + 21) : 10'' :: 50 : 7''$: of course, $41^{\circ} 48' 37''$ is another arc, of which the sum of the log. cosine and half the log. sine is equal to 19.7843181, and the log. sine of this arc, increased by the log. of 3, is the logarithm of 1.999999, another value of x , and which errs but by unity in the seventh place.

The third root, as it is generally called, of this equation, which is necessarily negative, and equal to the sum of the other two, belongs properly to the equation which is given as the first example, of which it is the affirmative root, and may be found by the directions which are there given.

E X A M P L E III.

To find the value of x in an equation of the form $x^3 + r^2x = a$.

Let us take as examples of this equation $x^3 + 3x = .04$, $x^3 + 3x = .08$, and $x^3 + 3x = .12$, which are three of the instances given by Dr. HALLEY, in his Synopsis of the Astronomy of Comets, to illustrate the mode of computation that he pursued in constructing his general table for calculating the place of a comet in a parabolic orbit: and it is obvious, a being put for the known side of the equation, that it may be transformed to $\sqrt{x} \times \sqrt{3+x^2} = \sqrt{a}$: where, if x be considered as the tangent of an arc, the radius of which is $\sqrt{3}$, $\sqrt{3+x^2}$ will be the secant of that arc; and, consequently, by what is shewn in the first example, an arc must be found, such, that the sum of the tabular log. secant and half the tabular log. tangent may be equal to the excess of half the log. of a above $\frac{1}{2}$ of the log. of 3. In the first of the above three instances this excess will be found, 18.9431891, in the second 19.0937041, and in the third 19.1817497; and by running the eye along GARDINER'S Tables of Logarithmic Sines and Tangents, it will be found, that the first falls between $0^\circ 26' 20''$ and $0^\circ 26' 30''$, the second between $0^\circ 52' 50''$ and $0^\circ 53' 0''$, and the third between $1^\circ 19' 20''$ and $1^\circ 19' 30''$; and, by pursuing the mode which has been described in the two former examples, the exact arcs will be found $0^\circ 26' 27''.7$, $0^\circ 54' 51''.7$, and $1^\circ 19' 20''.1$, and their respective tangents, to the radius $\sqrt{3}$, .01333248, .0266611, and .0399787, the three values of x sought. And in this manner Dr. HALLEY'S table may be extended to any length with the utmost ease, expedition, and accuracy.

Thus far this matter has been carried by former writers; but those who may be at the trouble of consulting them will find that I have not copied their methods: on the contrary, these which are given here are more plain and obvious than theirs are, and the operations considerably shorter. What follows has not, I believe, been adverted to by any before me.

E X A M P L E IV.

Let the equation arising from the proportion $a : b + x \cdot \sqrt{1 - c^2} :: c\sqrt{1 - x^2} : c^2x$ be taken, which is the result of an inquiry into the situation of that place on the surface of the earth, considered as a spheroid, which is at the greatest distance from a given one, suppose London. In this inquiry a and b were put to represent the sine and cosine of the latitude of the given place (in the spheroid); c for $\frac{229}{230}$, the ratio of the axes; and x for the sine of the distance of the required place from the opposite pole (in the spheroid also). The equation, which is of four dimensions with all the terms, is manifestly $accx = \frac{b + x \cdot \sqrt{1 - c^2}}{\sqrt{1 - x^2}} \times \sqrt{1 - x^2}$, or $\frac{x}{\sqrt{1 - x^2}} - x \cdot \frac{1 - c^2}{ac} = \frac{b}{ac}$; in which it is evident from tab. I. that the difference between the tangent and the product of the sine into a given quantity is known. In order, therefore, to find the value of x , compute $\frac{b}{ac}$, and $\frac{1 - c^2}{ac}$, and find the logarithm of the latter. Now, because the elliptic meridian differs but little from a circle, the place sought will not be far from the antipodes of the given one, and its distance from the opposite pole may therefore be estimated at $39^\circ 5'$;

39° 5'; and, having taken out the natural tangent, and logarithmic sine of this arc, add the logarithm of $\frac{1-c^2}{ac}$ to the latter, and find the number corresponding to the sum, which will be less than the natural tangent of 39° 5' by 2869. As this assumption is so near, take 39' 6'' for the next, repeat the operation, and the result will be 1935 too great. Then 4804 (2869 + 1935) : 60'' :: 2869 : 36''; which being added to 39° 5', gives 39° 5' 36'', for the co-latitude of the place sought, and the natural sine of this arc, or .6305856 is the value of x in this equation.

E X A M P L E V.

Let the equation $x^3 + \frac{b^2 - 2a^2}{4a}x^2 + \frac{2a^2 - b^2}{2}x - \frac{ab^2}{4} = 0$, be taken, which results from a solution of one case of the problem *de inclinationibus* of APOLLONIUS; but which, as it naturally rises to a solid problem, was not, I conceive, considered by that celebrated author. The result of the analysis, before any reduction takes place, is this proportion, $x + a : x - a :: 2\sqrt{ax} : b$; and hence, $\frac{x-a}{x+a}\sqrt{ax} = \frac{1}{2}b$. But it is here manifest, that if a be taken for the tangent of an arc, of which the radius is \sqrt{ax} , x will be the cotangent of it, and $\frac{x-a}{x+a}\sqrt{ax}$ ($= \frac{1}{2}b$) the cosine of twice that arc. Consequently, we have to find an arc, the tangent of which is to the cosine of twice that arc as a is to $\frac{1}{2}b$; and this being done, the natural co-tangent of that arc, to the proper radius, will be the value of x .

Thus, let a be 10, and b 24; and the difference of the logarithms of a and $\frac{1}{2}b$ will be 0.0791812. Now, by running the eye along GARDINER's Tables of logarithmic Sines and Tangents, it will be readily seen, that the log. tangent of $26^{\circ} 33' 50''$, when increased by 0.0791812, is less than the cosine of $53^{\circ} 7' 40''$ twice that arc; and that the log. tangent of $26^{\circ} 34' 0''$, when increased by the same quantity, is too great. And, by actually taking out the logarithms, and making the additions, the former will be found too small by 455, and the latter too great by 632. Then, $1087 (455 + 632) : 10'' :: 455 : 4''$; which being added to $26^{\circ} 33' 50''$ gives $26^{\circ} 33' 54''$ for the arc of which x is the co-tangent. And if to twice the log. co-tangent of this arc the logarithm of a (10) be added, the sum (1.6020600) is the log. of 40, the value of x sought.

E X A M P L E VI.

The equation resulting from a solution of the famous problem of ALHAZEN may be given as another example of the use of this method. Many solutions of this celebrated problem, by HUYGENS, SLUSIUS, and others, may be met with in the Philosophical Transactions. Solutions to it may also be found at the end of Dr. ROBERT SIMPSON's Conic Sections, in Dr. SMITH's Optics, Mr. ROBIN's Mathematical Tracts, and other places; but the most direct and obvious method is, perhaps, that which follows.

Put $a = DC$, $b = dC$, $r = CI$; $x = CB$ and $y = CE$, the cosines of the arcs IA , IH , to the radius r : then will the sines of those arcs, BA , EH , be expressed by $\sqrt{r^2 - x^2}$ and $\sqrt{r^2 - y^2}$; and,

and, because of the similar triangles ABC and BFC, HEC and dGC,

$$r : x :: a : \frac{ax}{r} = CE, r : \sqrt{r^2 - x^2} :: a :$$

$$\frac{a\sqrt{r^2 - x^2}}{r} = DF; r : y :: b : \frac{by}{r} = CG, \text{ and}$$

$$r : \sqrt{r^2 - y^2} :: b : \frac{b\sqrt{r^2 - y^2}}{r} = dG; \text{ con-}$$

sequently, $\frac{ax}{r} - r = FI, \frac{bx}{r} - r = GI$; and

because the angles DIF, dIG, are equal by the nature of the problem, and the angles DFI and dGI both right angles, the triangles DFI and dGI are also similar, and consequently $\frac{a\sqrt{r^2 - x^2}}{r} : \frac{ax}{r} - r ::$

$$\frac{b\sqrt{r^2 - y^2}}{r} : \frac{by}{r} - r; \text{ and } \frac{x}{\sqrt{r^2 - x^2}} - \frac{r^2}{a\sqrt{r^2 - x^2}} = \frac{y}{\sqrt{r^2 - y^2}} - \frac{r^2}{b\sqrt{r^2 - y^2}}; \text{ or}$$

$$\frac{r^2}{b\sqrt{r^2 - y^2}} - \frac{r^2}{a\sqrt{r^2 - x^2}} = \frac{y}{\sqrt{r^2 - y^2}} - \frac{x}{\sqrt{r^2 - x^2}}; \text{ or, the co-secant of the}$$

arc HI $\div b$ - co-secant of AI $\div a =$ the co-tangent HI $\div r$ - co-tangent AI $\div r$; or, lastly, the co-tangent of HI - co-tangent of AI = co-secant of HI $\times \frac{r}{b}$ - co-secant AI $\times \frac{r}{a}$. Consequently,

we have to find two arcs, the sum of which is given, and such that the difference of their co-tangents may be equal to the difference of the products of their co-secants into given quantities.

To do this assume the angle DCF as near as possible; and, because the sum of the two angles is given, the angle dCG will be known also. Take the difference of the logarithms of r and a , r and b , which will be constant, also the difference of the co-tangents of the two assumed arcs, and having taken out the log. co-secants, add to them respectively the two logarithmic

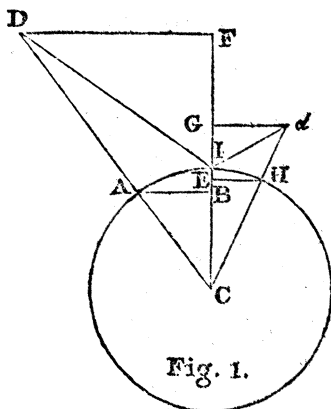


Fig. 1.

differences. Find the numbers corresponding to these two sums, and if the difference of these two numbers be equal to the difference of the co-tangents, the angle DCF was rightly assumed; but as that will seldom happen, take the difference, or error; assume the angle DCF again, repeat the operation, and find the error as before. Then, as the sum of the errors, if one of them was too great, and the other too little, or their difference, if both were too great, or both too little, is to the difference of these assumptions, so is the less error to a number of minutes and seconds, which must be added to that assumption to which the least error belongs, if that assumption was too small; or subtracted from it, if the assumption was too great: and, unless the first assumption was made very wide of the truth, which may always be avoided, the two angles will generally be obtained within a few seconds of the truth, and, by repeating the operation once more, to the utmost exactness.

Suppose DC (a) be taken equal to 72, dC (b) = 48, and the radius CI (r) = 40, the angle DC d being $82^{\circ} 45'$: then the whole operation will stand as follows:

$r = 40$ log.	11.6020600	-	-	11.6020600
$a = 72$ log.	1.8573325		$b = 48$ log.	1.6812412
	9.7446275		Constant log.	9.9208188

Now, in the two triangles DCI, dCI , the angles DIC and dIC being equal, and CI common, but dC considerably less than DC, it is manifest, that the angle dCI will be considerably less than the angle DCI: let them be assumed in the proportion that DC bears to its excess above dC ; in which case the angle dCI will be $27^{\circ} 35'$ and DCI $55^{\circ} 10'$. The co-tangent of the former will be 1.9141795, of the latter .6958813; and the difference of them 1.2182982. The log. co-secants of those

two

two angles are 10.3343832 and 10.0857536 , which being respectively increased by 9.9208188 and 9.7446275 , the two constant logarithms, make 0.2552020 and 9.8304811 , which are the logarithms of 1.7997079 and $.6768323$; and the difference of these two numbers is 1.1228756 , which is less than the difference of the log. co-tangents by $.0954226$.

I next assume the angles 30° and $52^\circ 45'$; and by pursuing the same steps which have been described above, I find the difference of their co-tangents exceeds the difference of the products by $.0028987$. Then, as 925239 (the difference of the errors) is to $145'$ (the difference of suppositions), so is the latter error 28987 to $4' 33''$, which being added to 30° , gives $30^\circ 4' 33''$ for the next assumption of the angle dCI ; but for ease in the computation I shall take $30^\circ 5'$; in which case the angle DCI will be $52^\circ 40'$; and by repeating the operation the difference of the co-tangents will be found less than the difference of the products by $.0002425$. And 31412 (the sum of the two last errors) is to $5'$ (the difference of the suppositions) as 2425 (the last error) is to $23''$; which being taken from $30^\circ 5'$, the last supposition, because it was too great, leaves $30^\circ 4' 37''$ for the exact value of the angle dCI .

This equation, like that in the fourth example, when the value of y is properly substituted, and the equation reduced in the usual manner, will rise to four dimensions with all the inferior ones; and it does not appear, that either HUYGENS, SLUSIUS, Mr. ROBINS, Dr. WILLSON, or Professor SIMSON, with all their artifice, have been able to depress it: but by this method of resolution the point of reflection is found, with the greatest exactness, in much less time than this substitution and reduction can be made. And this example farther suggests to us, that when the answer is sought by the method now under

consideration, it is not always necessary to exterminate all the unknown quantities but one.

E X A M P L E VII.

Suppose the equation to be resolved were $a = .375 = 16y^5 - 4y^4 - 20y^3 = 4y^2 + 5y$: and, first, let the upper signs have place, and it is manifest, that the latter side of the equation may be divided into two parts; namely, $4y^2 - 4y^4 = 4y^2 \cdot \overline{1+y} \cdot \overline{1-y}$, and $16y^5 - 20y^3 + 5y = y^5 - 10y^3 \cdot \overline{1+y} \cdot \overline{1-y} + 5y \cdot \overline{1+y}^2 \cdot \overline{1-y}^2$. But the former part is (by tab. I.) the square of the sine of twice the arc which has y for its sine (radius being = 1) and the latter part the sine of five times the same arc. Hence, therefore, the given quantity $a (= .375)$ is equal to the sum of the sine of five times the arc (A) which has y for its sine, and the square of the sine of twice the same arc. Now, as the square of the sine of twice the arc (A) must necessarily in this instance be very small in comparison of the sine of five times the same arc (A), it is manifest, that the sine of five times the arc which has y for its sine will be very little less than .375, and of course that arc (5A) can be but very little less than $22^\circ 2'$, the sine of which is next greater than that number. Assume it 21° , and the fifth part of it, or that arc which has y for its sine, will be $4^\circ 12'$, the double of which is $8^\circ 24'$. Now the log. sine of $8^\circ 24'$ is 9.1645998, which being doubled is 8.3291996, the logarithm of .0213403, and this number being taken from .375 leaves .3536597, which ought to have been .3583679, the sine of 21° , and of course is too small by .0047082: the arc has, therefore, been assumed too great.

Let

Let $20^{\circ} 45'$ be next aſſumed; the fifth part of which is $4^{\circ} 9'$, and twice this laſt number is $8^{\circ} 18'$, of which the log. ſine is 9.1594354; and this being doubled is 8.3188708, the log. of .0208387; and this being taken from .375 will leave .3541613: leſs ſtill than the ſine of $20^{\circ} 45'$ by .0001297.

Take now $20^{\circ} 44'$, the fifth of which is $4^{\circ} 8' 48''$, and two-fifths is $8^{\circ} 17' 36''$; and the log. ſine of this is 9.1590889, which being doubled gives 8.3181778, the logarithm of .0208055; and this being taken from .375, leaves .3541945; more than the ſine $20^{\circ} 44'$ by .0000795. Now 2092 (the ſum of the laſt two laſt errors) is to $60''$ as 795 (the laſt error) is to $23''$. Which being added to $20^{\circ} 44'$, the laſt aſſumption, gives $20^{\circ} 44' 23''$ for five times the arc of which y is the ſine: y is therefore the ſine of $4^{\circ} 8' 52''.6$, or .07233202.

When the lower ſigns in the equation have place, the given quantity a will be equal to the exceſs of the ſine of five times an arc above the ſquare of the ſine of twice that arc: and the operation, after aſſuming, from the circumſtances of the queſtion, or equation, an arc which is nearly five times that having y for its ſine, is this. Find the logarithmic ſine of two-fifths of that arc, double it, find the number correſponding to this logarithm, and add to it the value of a , which ſhould then be equal to the ſine of the arc firſt aſſumed; and if it is not, to repeat the operation until an error is obtained on each ſide, and not very diſtant from the truth, as is done above, and which may always be done with three aſſumptions.

A multitude of examples might be added from the writings of different authors, who have either left their concluſions

unexhibited in numbers, for want of some such easy method as this, or have done it by means of a long and laborious series of difficult computations; which, beside the labour attending them, are always subject to a variety of errors, which cannot be detected, in many cases, without repeating the operation.



T A B L E I.

Arc.	Sine.	Cofine.	Tangent.
A	x	$\sqrt{r+x, r-x}$	$\frac{rx}{\sqrt{r+x, r-x}}$
2A	$\frac{2x}{r} \cdot \sqrt{r+x, r-x}$	$\frac{r+x, r-x-x^2}{r}$	$\frac{2}{r+x, r-x-x^2} \sqrt{\frac{rx}{r+x, r-x}}$
3A	$\frac{3x \cdot r+x, r-x-x^2}{r^2}$	$\frac{r+x, r-x-3x^2}{r^2} \sqrt{r+x, r-x}$	$\frac{3 \cdot r+x, r-x-x^2}{r+x, r-x-3x^2} \times \frac{rx}{\sqrt{r+x, r-x}}$
4A	$\frac{4x \cdot r+x, r-x-4x^2}{r^3} \sqrt{r+x, r-x}$	$\frac{r+x, r-x-6x^2 \cdot r+x, r-x+x^4}{r^3}$	$\frac{4 \cdot r+x, r-x-4x^2}{r+x, r-x-6x^2 \cdot r+x, r-x+x^4} \sqrt{\frac{rx}{r+x, r-x}}$
5A	$\frac{5x \cdot r+x, r-x-10x^2 \cdot r+x, r-x+x^4}{r^4}$	$\frac{r+x, r-x-10x^2 \cdot r+x, r-x+5x^4}{r^4} \sqrt{r+x, r-x}$	$\frac{5 \cdot r+x, r-x-10x^2 \cdot r+x, r-x+x^4}{r+x, r-x-10x^2 \cdot r+x, r-x+5x^4} \times \frac{rx}{\sqrt{r+x, r-x}}$
6A	$\frac{6x \cdot r+x, r-x-20x^2 \cdot r+x, r-x+6x^4}{r^5} \sqrt{r+x, r-x}$	$\frac{r+x, r-x-15x^2 \cdot r+x, r-x+15x^4 \cdot r+x, r-x-x^6}{r^5}$	$\frac{6 \cdot r+x, r-x-20x^2 \cdot r+x, r-x+6x^4}{r+x, r-x-15x^2 \cdot r+x, r-x+15x^4 \cdot r+x, r-x} \sqrt{\frac{rx}{r+x, r-x}}$

T A B L E II.

Arc.	Sine.	Cofine.	Tangent.
A	$\frac{rx}{\sqrt{r^2+x^2}}$	$r \cdot \frac{r}{\sqrt{r^2+x^2}}$	x
2A	$\frac{2r^2x}{r^2+x^2}$	$r \cdot \frac{r+x, r-x}{r^2+x^2}$	$\frac{2r^2x}{r+x, r-x}$
3A	$\frac{r+x, r-x+2r^2}{r^2+x^2} \times \frac{rx}{\sqrt{r^2+x^2}}$	$\frac{r \cdot r+x, r-x-2rx}{r^2+x^2} \times \frac{r}{\sqrt{r^2+x^2}}$	$\frac{x \cdot r+x, r-x+2r^2x}{r+x, r-x-2x^2}$
4A	$\frac{2 \cdot r+x, r-x}{r^2+x^2} \times \frac{2r^2x}{r^2+x^2}$	$r \cdot \frac{r+x, r-x-2rx}{r^2+x^2} \times \frac{r+x, r-x+2rx}{r^2+x^2}$	$\frac{4r^2x \cdot r+x, r-x}{r+x, r-x-4r^2x^2}$
5A	$\frac{r+x, r-x+4r^2 \cdot r+x, r-x-4r^2x^2}{r^2+x^2 \cdot r^2+x^2} \times \frac{rx}{\sqrt{r^2+x^2}}$	$r \cdot \frac{r+x, r-x-4r^2 \cdot r+x, r-x-4r^2x^2}{r^2+x^2 \cdot r^2+x^2} \times \frac{r}{\sqrt{r^2+x^2}}$	$\frac{x \cdot r+x, r-x+4r^2x \cdot r+x, r-x-4r^2x^2}{r+x, r-x-4x^2 \cdot r+x, r-x-4r^2x^2}$
6A	$\frac{3 \cdot r+x, r-x-4r^2x^2}{r^2+x^2 \cdot r^2+x^2} \times \frac{2 \cdot r^2x}{r^2+x^2}$	$\frac{r+x, r-x-12r^2x^2}{r^2+x^2} \times \frac{r+x, r-x}{r^2+x^2} \times \frac{r}{r^2+x^2}$	$r^2 \cdot \frac{6x \cdot r+x, r-x-8r^2x^2}{r+x, r-x-12r^2x^2 \cdot r+x, r-x}$

T A B L E III.

Arc.	Sine.	Cofine.	Tangent.
A	$\frac{r \cdot \sqrt{x+r} \cdot x-r}{x}$	$r \cdot \frac{r}{x}$	$\frac{\sqrt{x+r} \cdot x-r}{x}$
2A	$r \cdot \frac{2r \cdot \sqrt{x+r} \cdot x-r}{x^2}$	$r \cdot \frac{r^2 - x+r \cdot x-r}{x^2}$	$\frac{2r^2 \cdot \sqrt{x+r} \cdot x-r}{r^2 - x+r \cdot x-r}$
3A	$r \cdot \frac{3r^2 - x+r \cdot x-r}{x^3} \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^3 - 3r \cdot x+r \cdot x-r}{x^3}$	$\frac{3r^2 - x+r \cdot x-r}{r - 3 \cdot x+r \cdot x-r} \sqrt{x+r} \cdot x-r$
4A	$\frac{r \cdot 4r^3 - 4r \cdot x+r \cdot x-r}{x^4} \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^4 - 6r^2 \cdot x+r \cdot x-r + x+r}{x^4} \cdot x-r$	$\frac{4r^3 - 4r^2 \cdot x+r \cdot x-r}{r^4 - 6r^2 \cdot x+r \cdot x-r + x+r} \sqrt{x+r} \cdot x-r$
5A	$r \cdot \frac{5r^4 - 20r^2 \cdot x+r \cdot x-r + x+r}{x^5} \cdot x-r \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^5 - 10r^3 \cdot x+r \cdot x-r + 5r \cdot x+r}{x^5} \cdot x-r$	$\frac{5r^4 - 10r^2 \cdot x+r \cdot x-r + x+r}{r^5 - 10r^2 \cdot x+r \cdot x-r + 5 \cdot x+r} \sqrt{x+r} \cdot x-r$
6A	$r \cdot \frac{6r^5 - 20r^3 \cdot x+r \cdot x-r + 6r \cdot x+r}{x^6} \cdot x-r \sqrt{x+r} \cdot x-r$	$r \cdot \frac{r^6 - 15r^4 \cdot x+r \cdot x-r + 15r^2 \cdot x+r}{x^6} \cdot x-r$	$\frac{6r^5 - 20r^3 \cdot x+r \cdot x-r + 6r^2 \cdot x+r}{r^6 - 15r^4 \cdot x+r \cdot x-r + 15r^2 \cdot x+r} \sqrt{x+r} \cdot x-r$

T A B L E IV.

Arc.	Sine.	Cofine.	Tangent.
A	$\sqrt{r^2 - r-x}$	$r-x$	$\frac{r}{r-x} \sqrt{r^2 - r-x}$
2A	$\frac{2 \cdot r-x}{r} \sqrt{r^2 - r-x}$	$\frac{2 \cdot r-x}{r} - r^2$	$\frac{2r \cdot r-x}{2 \cdot r-x - r^2} \sqrt{r^2 - r-x}$
3A	$\frac{4 \cdot r-x}{r^2} \sqrt{r^2 - r-x}$	$\frac{4 \cdot r-x}{r^2} - 3r^2 \cdot r-x$	$\frac{4r \cdot r-x - r^3}{4 \cdot r-x - 3r^2 \cdot r-x} \sqrt{r^2 - r-x}$
4A	$\frac{8 \cdot r-x}{r^3} \sqrt{r^2 - r-x}$	$\frac{8 \cdot r-x}{r^3} - 8r^2 \cdot r-x + r^4$	$\frac{8r \cdot r-x - 4r^2 \cdot r-x}{8 \cdot r-x - 8r^2 \cdot r-x + r^4} \sqrt{r^2 - r-x}$
5A	$\frac{16 \cdot r-x}{x^4} \sqrt{r^2 - r-x}$	$\frac{16 \cdot r-x}{r^4} - 20r^2 \cdot r-x + 5r^4 \cdot r-x$	$\frac{16r \cdot r-x - 12r^2 \cdot r-x + r^3}{16 \cdot r-x - 20r^2 \cdot r-x + 5r^4 \cdot r-x} \sqrt{r^2 - r-x}$
6A	$\frac{32 \cdot r-x}{r^5} \sqrt{r^2 - r-x}$	$\frac{32 \cdot r-x}{r^5} - 48r^2 \cdot r-x + 18r^4 \cdot r-x - r^6$	$\frac{32r \cdot r-x - 32r^2 \cdot r-x + 6r^4 \cdot r-x}{32 \cdot r-x - 48r^2 \cdot r-x + 18r^4 \cdot r-x - r^6} \sqrt{r^2 - r-x}$